Centrum Fizyki Teoretycznej PAN

Positive-P Simulation of Cooling Dynamics in a Trapped One-Dimensional Bose Gas

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Abstract

We study cooling dynamics of a one-dimensional trapped Bose gas using the Positive-P representation. Cooling is modeled as particle loss in an open quantum system, starting from a thermal state above the characteristic temperature. We analyze how cooling changes atom-number fluctuations, correlations, and condensate properties of the final gas.

Cooling Model

1D gas of interacting bosons in a harmonic trap

Continuous system as a Lattice Model

The continuous problem is represented on a lattice using the **Discrete Variable Representation (DVR)**. The DVR provides a set of localized basis functions associated with discrete spatial points.

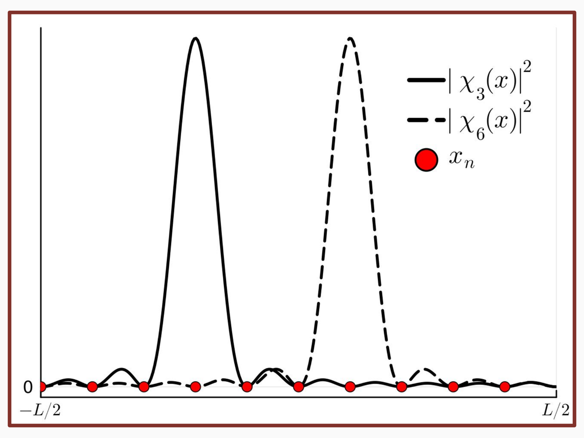

Fig. 1: Example density profile of DVR basis.

Hamiltonian

In the DVR basis, the kinetic energy is treated exactly, while the trapping and interaction potentials are represented on the grid. This leads to an effective **lattice Hamiltonian** for the trapped gas.

$$\hat{H} = \sum_{ij} K_{ij}\hat{a}_i^\dagger \hat{a}_j + \sum_i V(x_i)\hat{a}_i^\dagger \hat{a}_i + \sum_i \frac{g}{w_i}\hat{a}_i^\dagger \hat{a}_i^\dagger \hat{a}_i \hat{a}_i$$

Cooling Method

Cooling is modeled using a **master equation** with jump operators describing particle loss. Atoms are removed with a given rate when they move sufficiently far from the center of the trap.

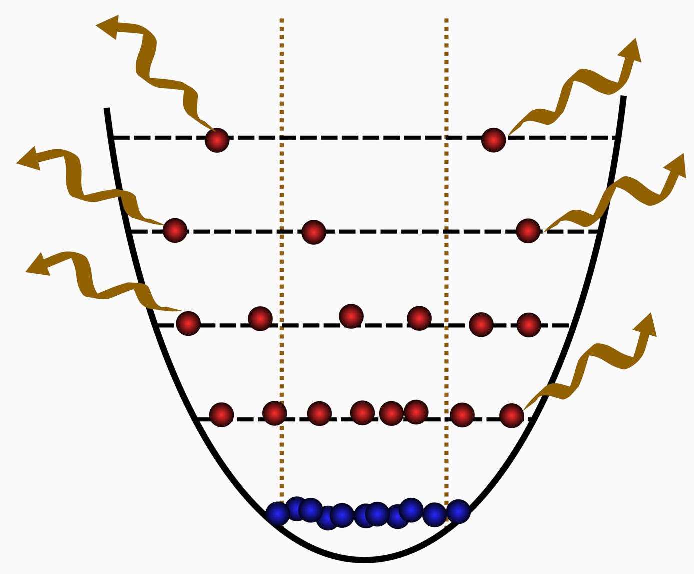

Fig. 2: Schematic of the cooling model. Particles are removed once they move beyond the cutoff position.

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H},\hat{\rho}\right] + \sum_j \left(\hat{\Gamma}_j \hat{\rho}\hat{\Gamma}_j^\dagger - \frac{1}{2}\left\{\hat{\Gamma}_j^\dagger \hat{\Gamma}_j, \hat{\rho}\right\}\right)$$

$$\hat{\Gamma}_j = \sqrt{\gamma_j}\hat{a}_j, \quad \text{where } \gamma_j = \begin{cases} \gamma & \text{if } x_j \geq x_{\text{cut}} \\ 0 & \text{otherwise} \end{cases}$$

Positive-P Simulation Method

Positive-P Representation

The density matrix is written in the **Positive-P representation** as a phase-space distribution over coherent-state projectors. Each lattice site is represented by a pair of independent complex variables.

$$\hat{\rho} = \int \mathrm{d}^2\boldsymbol{\alpha}\, \mathrm{d}^2\boldsymbol{\beta}\, P(\boldsymbol{\alpha},\boldsymbol{\beta},t)\hat{\Lambda}(\boldsymbol{\alpha},\boldsymbol{\beta})$$

$$\hat{\Lambda}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{|\boldsymbol{\alpha}\rangle\langle\boldsymbol{\beta}^*|}{\langle\boldsymbol{\beta}^*|\boldsymbol{\alpha}\rangle}$$

$$|\boldsymbol{\alpha}\rangle = \bigotimes_j |\alpha_j\rangle, \quad \hat{a}_j|\boldsymbol{\alpha}\rangle = \alpha_j|\boldsymbol{\alpha}\rangle$$

Initial State

Initial samples are generated from the **P-function** of a thermal state of noninteracting bosons in the canonical ensemble. The sampling is performed using variables associated with harmonic-oscillator modes.

$$P_{\text{GC}}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{1}{(2\pi)^{2M}}e^{-|\boldsymbol{\delta}|^2}\frac{1}{\Xi}\prod_{j=1}^{M}\exp\left[-\left(1-e^{-\beta(\epsilon_j-\mu)}\right)|\gamma_j|^2\right]$$

$$P_{\text{CN}}(\boldsymbol{\alpha},\boldsymbol{\beta}) = \frac{1}{(2\pi)^{2M}}e^{-|\boldsymbol{\delta}|^2}\frac{1}{Z_N N!}e^{-|\boldsymbol{\gamma}|^2}\left(\sum_{j=1}^{M}|\gamma_j|^2 e^{-\beta\epsilon_j}\right)^N$$

$$\text{where } \boldsymbol{\gamma} = \frac{\boldsymbol{\alpha}+\boldsymbol{\beta}^*}{2}, \quad \boldsymbol{\delta} = \frac{\boldsymbol{\alpha}-\boldsymbol{\beta}^*}{2}$$

Dynamics: Stochastic Equations

Using the Positive-P representation, the master equation is mapped onto **stochastic Itô equations**. An ensemble of initial samples is generated and then evolved in time.

Observables

The P-distribution is approximated by an ensemble of stochastic samples. Physical observables are calculated as averages over the corresponding trajectories.

$$\langle \hat{a}_i^{\dagger n}\hat{a}_j^m\rangle = \langle \beta_i^n \alpha_j^m\rangle_{\text{s}}, \quad \langle \hat{n}_j\rangle = \langle \beta_j \alpha_j\rangle_{\text{s}}$$

$$\begin{cases} \frac{d\alpha_j}{dt} &= -\frac{\gamma_j}{2}\alpha_j + iV(x_j)\alpha_j + i\sum_k K_{kj}\alpha_k - i\frac{g}{w_j}\alpha_j^2\beta_j + \sqrt{-i\frac{g}{w_j}}\alpha_j\xi_j(t) \\ \frac{d\beta_j}{dt} &= -\frac{\gamma_j}{2}\beta_j - iV(x_j)\beta_j - i\sum_k K_{kj}^*\beta_k + i\frac{g}{w_j}\beta_j^2\alpha_j + \sqrt{i\frac{g}{w_j}}\beta_j\eta_j(t) \end{cases}$$

Cooling Dynamics of the Interacting Gas

Time Evolution in Plots

The time evolution is presented as changing color, starting from the red for the initial conditions to the blue line – end of simulation.

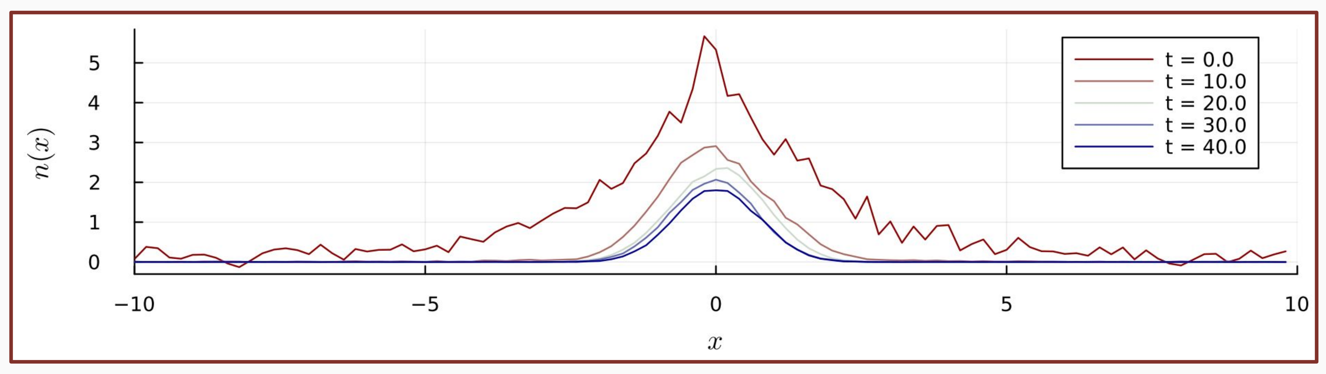

Fig. 3: Evolution of the density profile during cooling. As particles are removed, the cloud becomes narrower and more localized near the trap center.

Stability of the Simulation

The stochastic nature of the equation causes the simulation to become unstable after certain time. The dissipations stabilise this effect.

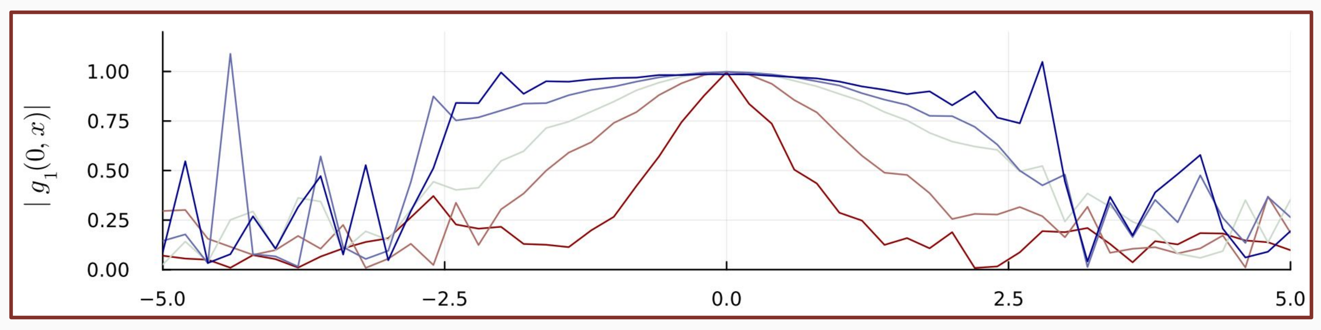

Fig. 4: Growth of the coherence length during cooling, indicating the formation of a more coherent condensed state.

Condensate fraction

The condensate fraction is estimated from the occupation of the dominant Gaussian-like mode.

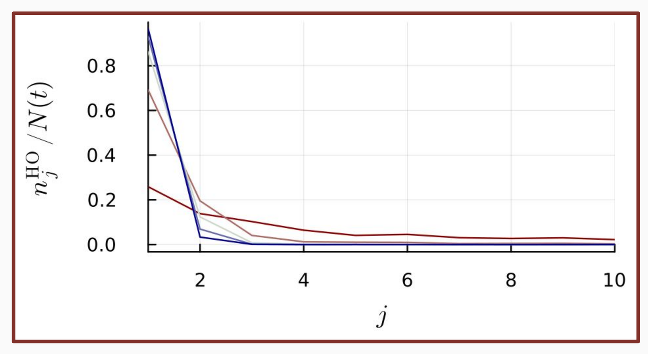

Fig. 5: Occupation of harmonic-oscillator modes during the cooling dynamics.

Temperature

The temperature is estimated by comparing the simulated atom distribution with the canonical-ensemble prediction. This is done by minimizing the **Kullback-Leibler divergence** (D(T)).

$$D(T) = -\sum_\nu n_\nu^{\text{sim}} \log\left(\frac{n_\nu^{\text{sim}}}{n_\nu^{\text{CN}}(T)}\right)$$

$$T_{\text{K}} = \underset{T}{\operatorname{argmin}} D(T)$$

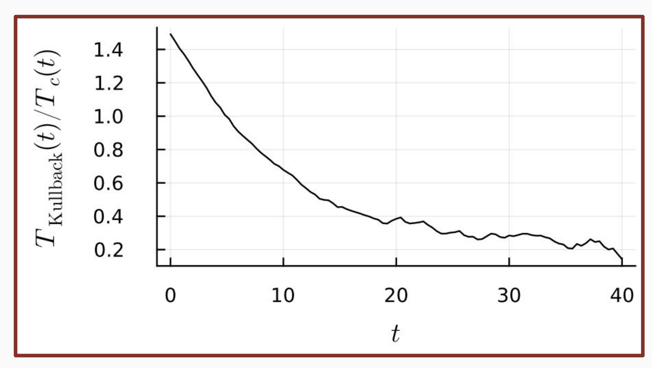

Fig. 6: Temperature estimated by minimizing the Kullback-Leibler divergence between the simulated distribution and the canonical-ensemble prediction.

Comparison with Statistical Ensembles

The estimated temperature lets us compare simulation results with statistical ensembles by plotting observables, such as mode occupation and fluctuations, against canonical and grand-canonical predictions.

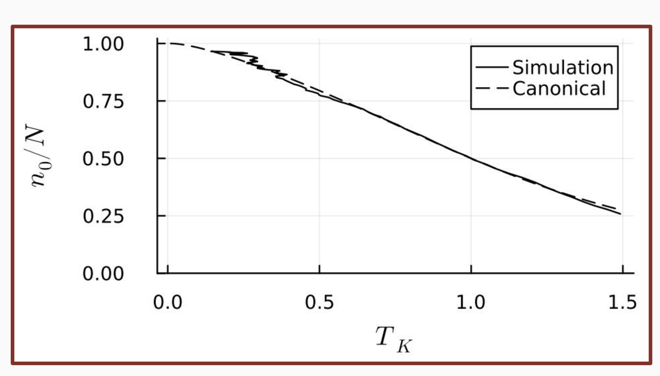

Fig. 7: Occupation of the Gaussian mode as a function of the inferred temperature.

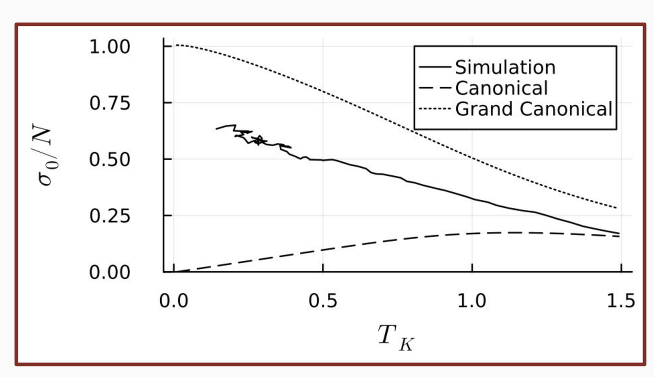

Fig. 8: Relative fluctuations of the Gaussian mode occupation as a function of the inferred temperature.

Conclusions

Does cooling lead to thermodynamic equilibrium? The mean occupation suggests thermalization, but the fluctuations do not match canonical or grand-canonical predictions. Thus, fluctuations reveal non-equilibrium features that are hidden in average quantities.

[1] P. Deuar and P. D. Drummond. Gauge P-representations for quantum-dynamical problems: Removal of boundary terms. Physical Review A, 66(3):033812, September 2002.
[2] P. D. Drummond and C. W. Gardiner. Generalised P-representations in quantum optics. Journal of Physics A: Mathematical and General, 1980.

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