Analytical results for the statistics of the ideal Bose gas in different statistical ensembles

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Introduction

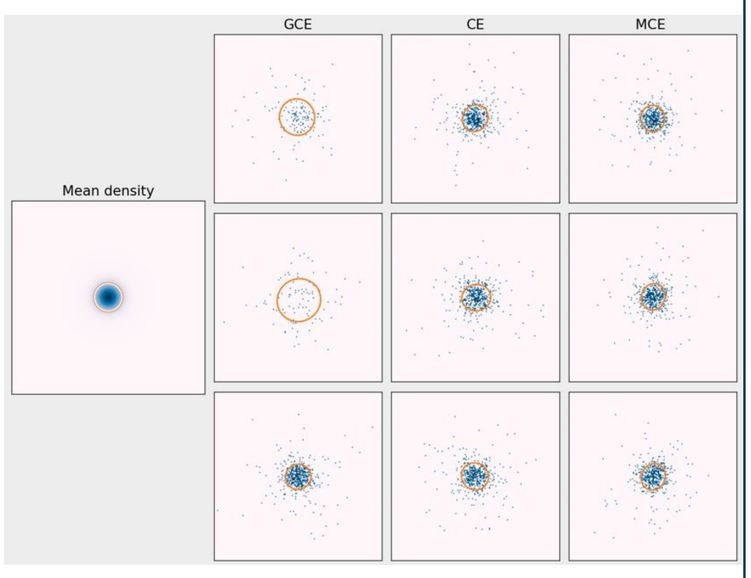
Statistical ensembles

- GCE (grand canonical ensemble): both E and N may fluctuate.
- CE (canonical ensemble): N is fixed, while E may fluctuate.
- MCE (microcanonical ensemble): both N and E are fixed.

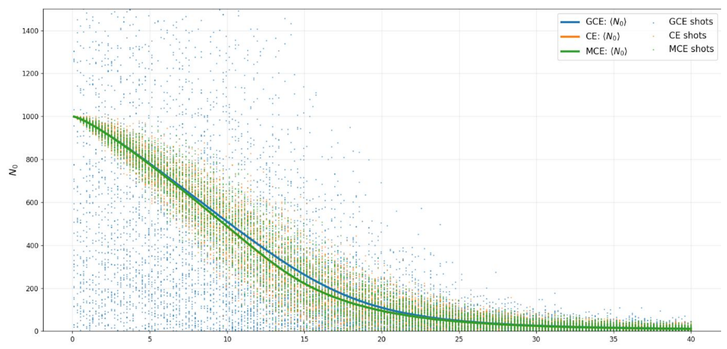
Relations for partition function in CE

- $\Xi(z,T) = \sum_{N=0}^{\infty} z^N Z(N,T)$
- $Z(N,T) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\Xi(z,T)}{z^{N+1}} \, \mathrm{d}z$

Motivation



Simulation of the measurement of cloud of atoms trapped in 2D harmonic trap, N = 500.
Different columns corresponds to different statistical ensembles, but with same (average) number of atoms at the same temperature.



Condensate population versus temperature, 1D harmonic trap, N = 1000

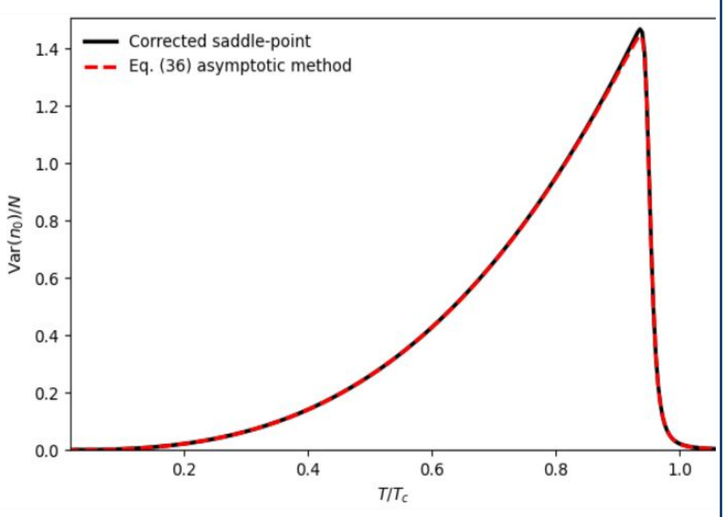
Main goals

Goals of this work

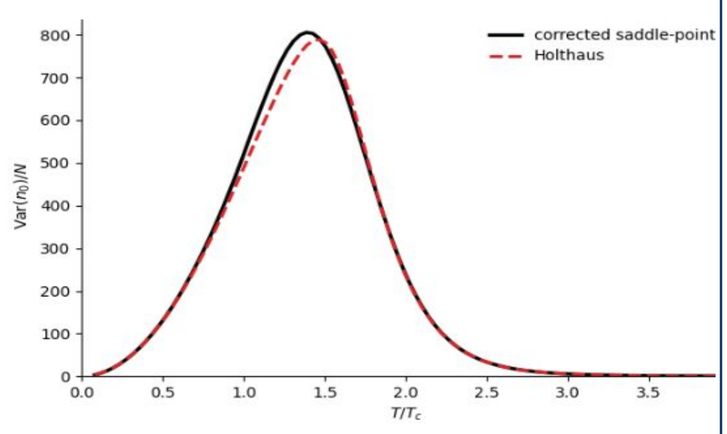
- Find a method better suited to 1D systems than a saddle-point method by Holthaus.
- To find method applicable to any non-interacting Bose systems in any ensemble.
- Derive the analytical formulas for the statistics.

Saddle-point method

Works pretty well for some systems:



For other systems it does not work:



Top: Var(n_0) vs temperature, 3D harmonic trap, CE, N = 5000.
Bottom: Var(n_0) vs temperature, 1D harmonic trap, CE, N = 10000.

Classification

BEC in systems like 3D trap in TL:

$$P(n_0 = n)_{\mathrm{CE}} \sim \exp\left[n w_0 - \frac{(n-n_0)^2}{2\kappa_2} \right]$$

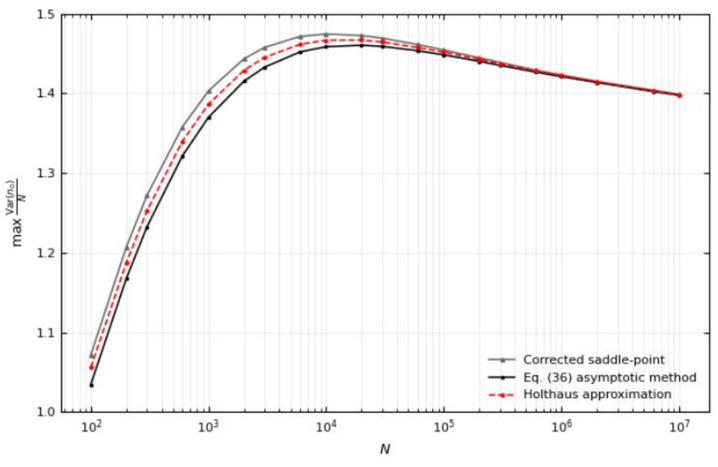
It works in the microcanonical ensemble as well:

$$P(n_0 = n)_{\mathrm{MCE}} \sim \exp\left[n w_0 - \frac{(n-n_0)^2}{2\widetilde{\kappa}_2} \right]$$

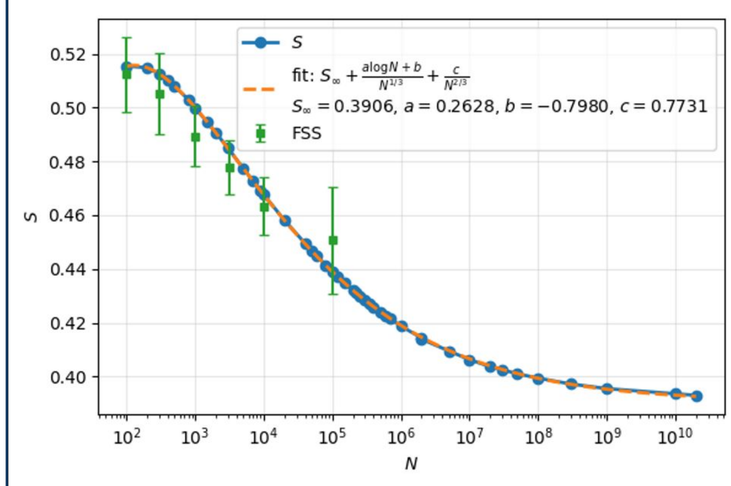
When does it work?

$$m \gg \max\left\{ \frac{\kappa_3}{\kappa_2^{3/2}}, m_R \right\}$$

How well does it work?



Practical results



Ratio of the maxima of fluctuations in a 3D harmonic trap in MCE and CE versus N

General formula in CE

All moments in general case :

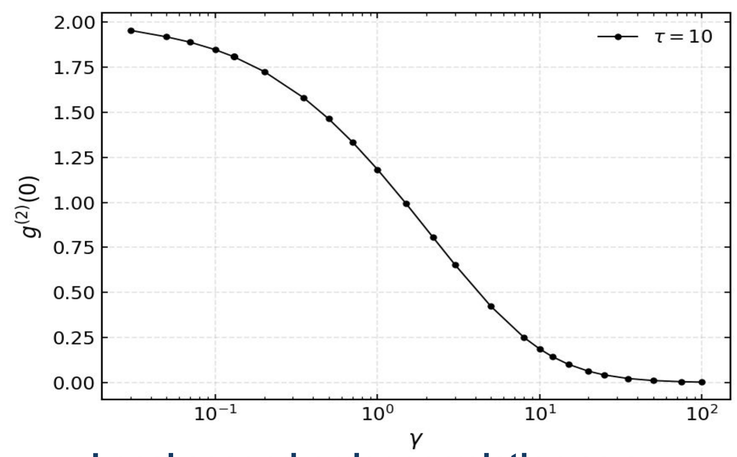
$$Z_N(u) = \mathrm{Re} \sum_{\substack{n > -n_0 \\ j \in \mathcal{J}_\alpha}} \frac{e^{(n+n_0)\,u}}{\pi\, \kappa_2^{1/2}} \int_0^{\pi/m} e^{\,\psi_{ex}^{(j)} + \left(n + \delta\kappa_1^{(j)}\right) w_0^{(j)}\left(\frac{v}{\sqrt{\kappa_2}}\right) - \frac{1}{2} v^2} \, \mathrm{d}v,$$

$$\langle n_0^m \rangle_{\mathrm{CE}} = \frac{1}{Z_N(0)} \, \partial_u^m Z_N(u)\big|_{u=0} .$$

Why is this formula useful?

- Asymptotically Reproduces the saddle-point approximation
- Allows to calculate N of order of millions
- Direct extension to microcanonical ensemble

Future generalizations



Local second-order correlation as a function of the interaction strength for a fixed reduced temperature

Future goals

- Generalize the saddle-point method approach to Yang-Yang model
- Study fluctuations of the condensate in different ensembles
- Employ generalized hydrodynamics to study time evolution and dissipation

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